DEPARTMENTS.

DISCUSSION.

THE EVALUATION OF $\int_0^\pi \frac{\sin mx}{x} dx$.

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The following solution of problem 203, Calculus, the evaluation of the definite integral $\int_0^\pi \frac{\sin mx}{x} dx$ (m an integer), involves several points of interest.

$$\int_0^\pi \frac{\sin mx}{x} dx = \int_0^{m\pi} \frac{\sin x}{x} dx.$$

Developing $\int \frac{\sin x}{x} dx$ by the formula,*

$$f(x)=f(0)+\frac{x}{r2}\left\{[f'(x)+f'(0)]+2\left[f'\left[\frac{x}{r}\right]+f'\left[\frac{2x}{r}\right]+f'\left[\frac{3x}{r}\right]+\cdots\cdots\right.\right.$$

$$\left.\left.+f'\left[\frac{r-1}{r}x\right]\right]\right\}-\frac{B_1 x^2}{r^2.2!}[f''(x)-f''(0)]+\frac{B_2 x^4}{r^4.4!}[f^{\text{iv}}(x)-f^{\text{iv}}(0)]$$

$$-\frac{B_3 x^6}{r^6.6!}[f^{\text{vi}}(x)-f^{\text{vi}}(0)]+\cdots\cdots+(-1)^n\frac{B_n\, x^{(2n)}}{r^{(2n)}.\,(2n)!}[f^{(2n)}(x)-f^{(2n)}(0)]+\cdots\cdots (1),$$

(B_1, B_2, B_3, ..., being Bernoulli's numbers), and taking $r=2m$, we get

$$\int\frac{\sin x}{x}dx=c+\frac{x}{(2m).2!}\left\{\left[\frac{\sin x}{x}+1\right]+2\left(\frac{\sin\left[\frac{x}{2m}\right]}{\frac{x}{2m}}+\frac{\sin\left[\frac{2x}{2m}\right]}{\frac{2x}{2m}}+\frac{\sin\left[\frac{3x}{2m}\right]}{\frac{3x}{2m}}\right.\right.$$

$$\left.\left.+\cdots\cdots+\frac{\sin\left[\frac{2m-1}{2m}x\right]}{\frac{(2m-1)x}{2m}}\right)\right\}-\frac{x^2}{6.(2m)^2.2!}\left[\frac{x\cos x-\sin x}{x^2}\right]$$

$$+\frac{x^2}{30.(2m)^4.4!}\left[\left(\frac{6-x^2}{x^3}\right)\cos x+\left(\frac{6-3x^2}{x^4}\right)\sin x\right]\cdots\cdots(2).$$

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But as m is an integer, $\int_0^{m\pi}\frac{\sin x}{x}dx$ develops by means of (2) into

$$\tfrac{1}{4}\pi+\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots\ldots\ldots-(-1)^m\frac{1}{(2m-1)}\right)\pm\pi\left(\frac{1}{6.2^2.m.2!}+\frac{(m\pi)^2-6}{30.2^4.m^3.4!}\right.$$

$$\left.+\frac{(m\pi)^4-20(m\pi)^2+120}{42.2^6.m^5.6!}+\ldots\ldots\ldots\right)\ldots\ldots\ldots(3),$$

according as m is odd or even.

For convenient use in numerical computation (3) may be put into the form

$$\int_0^{m\pi}\frac{\sin x}{x}dx=\tfrac{1}{4}\pi+\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots\ldots\ldots-(-1)^m\frac{1}{(2m-1)}\right)$$

$$\pm\left(\frac{c_1}{m}-\frac{c_3}{m^3}+\frac{c_5}{m^5}-\ldots\ldots\ldots\right)\ldots\ldots\ldots(4),$$

where $c_1=.0682995$, $c_3=.0019567$, $c_5=.0001948$, approximately.

By means of (4) the values of the definite integral corresponding to a few values of m are readily found to be as follows:

For $m=1$, 1.851936–
$m=2$, 1.418158+
$m=3$, 1.674760–
$m=4$, 1.492164+
$m=5$, 1.633963–
$m=6$, 1.518036–
....

SOLUTIONS OF PROBLEMS.

ALGEBRA.

247. Proposed by PROFESSOR G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Find the sum, to n terms, of

$$1+\frac{n}{2}+\frac{n(n+2)}{2.4}+\frac{n(n+2)(n+4)}{2.4.6}+\ldots\ldots\ldots$$

I. Solution by the PROPOSER.

The series is the coefficient of x^{n-1} in $(1-x)^{-\frac{1}{2}n}(1-x)^{-1}$; *i. e.*, in $(1-x)^{-(\frac{1}{2}n+1)}$. Hence the required sum is

$$\frac{(n+2)(n+4)\ldots\ldots(3n-2)}{2.4\ldots\ldots(2n-2)}.$$